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Entropy of Killing horizons from Virasoro algebra in D-dimensional extended Gauss–Bonnet gravity

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Abstract

We treat D-dimensional black holes with Killing horizon for extended Gauss–Bonnet gravity. We use the Carlip method and impose boundary conditions on the horizon what enables us to identify Virasoro algebra and evaluate its central charge and Hamiltonian eigenvalue. The Cardy formula allows then to calculate the number of states and thus provides for a microscopic interpretation of entropy.

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1. Introduction

The purpose of this Letter is to investigate how some recent results on microscopic interpretation of black hole entropy depend on the form of gravity action. The problem of microscopic description of black hole entropy was approached by different methods, like, e.g., string theory, which treated extremal and near extremal black holes or, e.g., loop quantum gravity (see references in [1]). Another line of approach to this problem is based on conformal field theory and Virasoro algebra. Such an algebra was identified by Brown and Henneaux [2] in $2+1$ dimensions and after requiring asymptotic AdS_3 symmetry. The well-known Bekenstein–Hawking entropy formula for Einstein gravity black holes was then reproduced [3]. In fact, there are essentially two independent approaches based on conformal field theory. One particular formulation was due to Solodukhin who reduced the problem of D-dimensional black holes to effective two-dimensional theory with fixed boundary conditions on horizon. This effective theory admits Virasoro algebra near horizon and calculation of its central charge allows to compute the entropy [4–7]. Another approach based on conformal field theory was developed by Carlip [8]. In fact Carlip has shown that under certain simple assumptions on boundary conditions near black hole horizon, one can identify a

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subalgebra of algebra of diffeomorphisms, which turns out to be Virasoro algebra. The fixed boundary conditions give rise to central extension of this algebra. The entropy is then calculated from Cardy formula [9]

$$S_{\mathrm{C}} = 2\pi\sqrt{\left(\frac{c}{6} - 4\Delta_g\right)\left(\Delta - \frac{c}{24}\right)}, \tag{1}$$

where Δ is the eigenvalue of Virasoro generator L_0 for the state we calculate the entropy and Δ_g is the smallest eigenvalue. The corresponding entropy reproduces the Bekenstein–Hawking formula. Till now, similar analysis was done for Einstein gravity and for dilaton gravity [8,10]. Here, we shall consider Gauss–Bonnet generalization of Einstein gravity in D dimensions. In fact it is known [11,12] that classical entropy differs generally from area law valid in Einstein theory and that for more general diffeomorphism invariant theory the entropy of black hole with bifurcate horizon is

$$S = -2\pi \int\limits_{\mathcal{H}} \hat{\boldsymbol{\epsilon}} E_R^{abcd} \eta_{ab} \eta_{cd}. \tag{2}$$

Here, $\mathcal{H}$ is a cross section of the horizon, η_{ab} denotes binormal to $\mathcal{H}$ and $\hat{\epsilon}$ is induced volume element on $\mathcal{H}$. The tensor E_R^{abcd} is given with

$$E_R^{abcd} = \frac{\partial L}{\partial R_{abcd}}. \tag{3}$$

The tensor E_R^{abcd} has all symmetries of Riemann tensor R^{abcd}. In this Letter we shall treat Gauss–Bonnet gravity with Lagrangian density

$$L = -\sum_{m=0}^{[D/2]} \lambda_m L_m(g). \tag{4}$$

Here, $[D]$ denotes integer part of D. The mth density is

$$L_m(g) = \frac{(-1)^m}{2^m} \delta^{c_1 d_1 \cdots c_m d_m}_{a_1 b_1 \cdots a_m b_m} R^{a_1 b_1}{}_{c_1 d_1} \cdots R^{a_m b_m}{}_{c_m d_m}, \tag{5}$$

where $\delta^{b_1 \cdots b_k}_{a_1 \cdots a_k}$ is totally antisymmetric product of k Kronecker deltas, normalized to take values 0 and ± 1. Corresponding tensor E_R^{abcd} reads

$$E_R{}^{cd}{}_{ab} = -\sum_{m=0}^{[D/2]} m\lambda_m \frac{(-1)^m}{2^m} \delta^{cd c_2 d_2 \cdots c_m d_m}_{ab a_2 b_2 \cdots a_m b_m} R^{a_2 b_2}{}_{c_2 d_2} \cdots R^{a_m b_m}{}_{c_m d_m}. \tag{6}$$

The problem of microscopic description for this case was treated with Solodukhin's method by us [6]. This method allows to obtain a relation between conformal charge and eigenvalue Δ but not their values independently. Consequently the relation between entropy derived with Cardy formula and classical entropy was a proportionality relation containing an unknown parameter. The method relied also essentially on particular assumptions like spherical symmetry.

Here we want to treat Gauss–Bonnet gravity with Carlip method, which is using Wald's covariant approach [12,14,15] and is more suitable to generalizations. We shall treat general black holes with Killing horizons without particular restrictions to spherical symmetry. We shall obtain separate values of conformal charge and eigenvalues of Hamiltonian. Also due to an interesting discussion about assumptions needed for these methods to be valid [16] and for these two methods to be consistent [5] one is motivated to test the method for different interactions. Indeed in the present derivation for Gauss–Bonnet gravity we find consistency with Solodukhin method when the latter is amended in the sense of Carlip proposal [5] as was done by us previously [6].

2. Horizon as boundary

We shall use covariant phase space approach developed for a general diffeomorphism invariant field theory [14, 15]. For a given vector field ξ^a defining a diffeomorphism, one can write corresponding Hamiltonian as a pure surface term

$$H[\xi] = \int_{\partial C} \big(\mathbf{Q}[\xi] - \xi \cdot \mathbf{B}\big) \tag{7}$$

provided that a $(D-1)$-form $\mathbf{B}$, defined with

$$\delta \int_{\partial C} \xi \cdot \mathbf{B} = \int_{\partial C} \xi \cdot \boldsymbol{\Theta}, \tag{8}$$

exists. Here $\mathbf{J} = d\mathbf{Q}$ and definitions of symplectic potential $\boldsymbol{\Theta}$ and conserved current $\mathbf{J}$ are given in [8]. Due to vanishing on shell of bulk terms variation of $H[\xi]$ is equal to variation of the boundary term $J[\xi]$. Following [2,8], one obtains for the Dirac bracket $\{J[\xi_1], J[\xi_2]\}^*$

$$\big\{J[\xi_1], J[\xi_2]\big\}^* = \int_{\partial C} \big(\xi_2 \cdot \boldsymbol{\Theta}[\phi, \mathcal{L}_{\xi_1}\phi] - \xi_1 \cdot \boldsymbol{\Theta}[\phi, \mathcal{L}_{\xi_2}\phi] - \xi_2 \cdot (\xi_1 \cdot \mathbf{L})\big), \tag{9}$$

and the algebra

$$\big\{J[\xi_1], J[\xi_2]\big\}^* = J\big[\{\xi_1, \xi_2\}\big] + K[\xi_1, \xi_2] \tag{10}$$

with central extension K. Due to Bianchi identity and antisymmetric properties of δ symbol in (6), one finds for Gauss–Bonnet case

$$\nabla_d E_R^{abcd} = 0. \tag{11}$$

Thus symplectic potential [12] takes simple form

$$\Theta_{pa_1\cdots a_{n-2}} = 2\epsilon_{apa_1\cdots a_{n-2}} E_R^{abcd} \nabla_d \delta g_{bc} \tag{12}$$

and the special form of (9) for Gauss–Bonnet case is

$$\big\{J[\xi_1], J[\xi_2]\big\}^* = 2\int_{\partial C} \big\{\epsilon_{apa_1\cdots a_{n-2}} \big(\xi_2^p E_R^{abcd} \nabla_d \delta_1 g_{bc} - \xi_1^p E_R^{abcd} \nabla_d \delta_2 g_{bc}\big) - \xi_2 \cdot (\xi_1 \cdot \mathbf{L})\big\}. \tag{13}$$

We shall now impose existence of Killing horizon and consider a certain class of boundary conditions on it [8]. In particular we assume D-dimensional spacetime M with boundary ∂M such that we have a Killing vector χ^a

$$\chi^2 = g_{ab}\chi^a\chi^b = 0 \quad \text{at } \partial M. \tag{14}$$

Near the horizon ("stretched horizon") we define ρ_a

$$\nabla_a \chi^2 = -2\kappa\rho_a. \tag{15}$$

Variations are required to satisfy boundary conditions near the horizon as follows

$$\begin{gathered} \frac{\chi^a\chi^b}{\chi^2}\delta g_{ab} \to 0, \qquad \chi^a t^b \delta g_{ab} \to 0, \\ \rho^a \nabla_a \big(g_{bc}\delta g^{bc}\big) = 0, \qquad \rho^a \nabla_a \left(\frac{\rho^b \delta\chi_b}{\chi^2}\right) = \rho^a \nabla_a \left(\frac{\delta\rho^2}{\rho^2}\right) = 0 \quad \text{at } \chi^2 = 0, \end{gathered} \tag{16}$$

and we keep χ^a and ρ_a fixed. Here t^a is any unit spacelike vector tangent to ∂M. We shall consider diffeomorphisms generated by vector fields ξ^a where

$$\xi^a = T\chi^a + R\rho^a, \tag{17}$$

with conditions

$$R = \frac{1}{\kappa}\frac{\chi^2}{\rho^2}\chi^a\nabla_a T, \qquad \rho^a\nabla_a T = 0. \tag{18}$$

An additional requirement will be necessary as already explained in [8]

$$\delta\int_{\partial C}\hat{\epsilon}\left(\tilde{\kappa} - \frac{\rho}{|\chi|}\kappa\right) = 0, \tag{19}$$

where $\tilde{\kappa}^2 = -a^2/\chi^2$, and $a^a = \chi^b\nabla_b\chi^a$ is the acceleration of an orbit of χ^a. This condition will guarantee existence of generators $H[\xi]$.

Now we want to calculate the central term from (10). In evaluating (13) we integrate over $(D-2)$-surface $\mathcal{H}$, which is the intersection of Killing horizon $\chi^2 = 0$ with the Cauchy surface C. As usual we introduce two null normals on $\mathcal{H}$. One is Killing vector χ^a and the other is future directed null normal $N^a = k^a - \alpha\chi^a - t^a$, where t^a is tangent to $\mathcal{H}$ and has a norm $t^2 = 2\alpha - \alpha^2\chi^2$, and $k^a = -(\chi^a - \rho^a|\chi|/\rho)/\chi^2$. Now the volume element can be written as

$$\epsilon_{bca_1\cdots a_{n-2}} = \hat{\epsilon}_{a_1\cdots a_{n-2}}\eta_{bc} + \cdots, \tag{20}$$

where only the first term contributes to the integral, and binormal η_{ab} is

$$\eta_{ab} = 2\chi_{[b}N_{c]} = \frac{2}{|\chi|\rho}\rho_{[a}\chi_{b]} + t_{[a}\chi_{b]}. \tag{21}$$

For the purpose of evaluation of integral (13) over the horizon we need to evaluate integrands to the lowest order in χ^2. We use

$$\nabla_d\delta g_{ab} \equiv \nabla_d\nabla_a\xi_b + \nabla_d\nabla_b\xi_a = -2\chi_d\chi_a\chi_b\frac{\ddot{T}}{\chi^4} + 2\chi_d\chi_{(a}\rho_{b)}\left(\frac{\dddot{T}}{\kappa\chi^2\rho^2} + \frac{2\kappa\dot{T}}{\chi^4}\right), \tag{22}$$

together with symmetries of E_R^{abcd} for first two terms, and finiteness of Lagrangian on the horizon for the third term. Finally, we get

$$\{J[\xi_1], J[\xi_2]\}^* = \frac{1}{2}\int_{\mathcal{H}}\hat{\epsilon}_{a_1\cdots a_{n-2}}E_R^{abcd}\eta_{ab}\eta_{cd}\left(\frac{1}{\kappa}\left(T_1\dddot{T}_2 - T_2\dddot{T}_1\right) - 2\kappa\left(T_1\dot{T}_2 - T_2\dot{T}_1\right)\right). \tag{23}$$

Next we need to calculate the Noether charge

$$Q_{c_3\cdots c_n} = -E_R^{abcd}\epsilon_{abc_3\cdots c_n}\nabla_{[c}\xi_{d]} = -\frac{1}{2}E_R^{abcd}\eta_{ab}\eta_{cd}\left(2\kappa T - \frac{\ddot{T}}{\kappa}\right)\hat{\epsilon}_{c_3\cdots c_n}. \tag{24}$$

Using the same method as in (23), we can calculate from (24) and (7)[1]

$$J\left[\{\xi_1,\xi_2\}\right] = -\frac{1}{2}\int_{\mathcal{H}}\hat{\epsilon}_{a_1\cdots a_{n-2}}E_R^{abcd}\eta_{ab}\eta_{cd}\left(2\kappa\left(T_1\dot{T}_2 - T_2\dot{T}_1\right) - \frac{1}{\kappa}\left(\dot{T}_1\ddot{T}_2 - \ddot{T}_1\dot{T}_2 + T_1\dddot{T}_2 - \dddot{T}_1T_2\right)\right). \tag{25}$$

[1] As in Einstein case the second term in (7) can be neglected: condition (19) enables us to factorize $\xi\cdot\boldsymbol{\Theta}$ into $\frac{1}{2}E_R^{abcd}\eta_{ab}\eta_{cd}\times\delta$ (terms that vanish on shell), which together with (8) implies that $\int_{\mathcal{H}}\xi\cdot\mathbf{B}$ vanishes on shell.

Now we are able to deduce central charge from (10), (23) and (25)

$$K[\xi_1, \xi_2] = -\frac{1}{2} \int_{\mathcal{H}} \hat{\epsilon}_{a_1 \cdots a_{n-2}} E_R^{abcd} \eta_{ab} \eta_{cd} \frac{1}{\kappa} \big(\dot{T}_1 \ddot{T}_2 - \ddot{T}_1 \dot{T}_2 \big). \tag{26}$$

3. Conformal charge and entropy

In previous sections we have introduced constraint algebra (10) where we have calculated various terms. As explained in [8], this algebra can be connected to the Virasoro algebra of diffeomorphisms of the circle or the real line provided we require the following condition

$$\frac{1}{A} \int_{\mathcal{H}} \hat{\epsilon}\, T_1(v, \theta) T_2(v, \theta) = \frac{\kappa'}{2\pi} \int dv\, T_1(v, \theta) T_2(v, \theta). \tag{27}$$

Here, v is the parameter of the orbits of the Killing vector $\chi^a \nabla_a v = 1$, θ denotes angular coordinates, $A \equiv \int_{\mathcal{H}} \hat{\epsilon}$ is the area of the horizon and $2\pi/\kappa'$ is period in the variable v of the functions $T(v, \theta)$. For rotating black hole

$$\chi^a = t^a + \sum \Omega_i \psi_i^a, \tag{28}$$

where t^a is time translation Killing vector, ψ_i^a are rotational Killing vectors with corresponding angles ψ_i and angular velocities Ω_i. The variables t, ψ_i associated with orbits of t^a, ψ_i^a, and variables (v, θ_i) associated with orbits of χ^a, $\theta_i^a = \psi_i^a$ are related with $v = t$, $\theta_i = \psi_i - \Omega_i v$. We choose for diffeomorphism defining functions T_n

$$T_n(v, \theta_i) = \frac{1}{\kappa'} e^{in(\kappa' v + \sum l_i \theta_i)}, \tag{29}$$

where l_i are integers. It can be checked that Lie brackets of corresponding diffeomorphisms satisfy classical Virasoro algebra. Also we see that condition (27) is fullfilled and thus enables us to obtain full Virasoro algebra with nontrivial central term $K[T_m, T_n]$ which can be calculated from (26)

$$i K[T_m, T_n] = \left(\frac{\kappa'}{\kappa} \right) \frac{\hat{A}}{8\pi} m^3 \delta_{m+n,0}, \tag{30}$$

where

$$\hat{A} \equiv -8\pi \int_{\mathcal{H}} \hat{\epsilon}_{a_1 \cdots a_{n-2}} E_R^{abcd} \eta_{ab} \eta_{cd}. \tag{31}$$

Here, we have used that metric does not depend on variables θ_i on which diffeomorphism defining functions T_n depend. That enabled us to factorize the integral in (26). Finally, we obtain Virasoro algebra

$$i \big\{ J[\xi_1], J[\xi_2] \big\}^* = (m - n) J[T_{m+n}] + \frac{c}{12} m^3 \delta_{m+n,0}, \tag{32}$$

with central charge c equal to

$$\frac{c}{12} = \frac{\hat{A}}{8\pi} \frac{\kappa'}{\kappa}. \tag{33}$$

From relation (24) we can calculate the eigenvalue of the Hamiltonian

$$\Delta \equiv J[T_0] = -\int_{\mathcal{H}} \hat{\epsilon}_{a_1 \cdots a_{n-2}} E_R^{abcd} \eta_{ab} \eta_{cd} \frac{\kappa}{\kappa'} = \frac{\kappa}{\kappa'} \hat{A}. \tag{34}$$

We are interested in calculating entropy via Cardy formula (1). Thus

$$S = \frac{\hat{A}}{4}\sqrt{2 - \left(\frac{\kappa'}{\kappa}\right)^2}. \tag{35}$$

Thus entropy is proportional to the classical entropy (2). The constant of proportionality is dimensionless. The proportionality relation becomes equality when we take for the period of functions T_n the period of the Euclidean black hole ([6,8,17] and references therein).

In that case we obtain

$$\frac{c}{12} = \Delta, \tag{36}$$

together with classical result (2) which can be also written in more explicit form using specific properties of Gauss–Bonnet gravity [18] as follows

$$S = -4\pi \sum_{m=1}^{[D/2]} m\lambda_m \int \hat{\epsilon} L_{m-1}. \tag{37}$$

4. Conclusion

In this Letter we have tried to make progress in the efforts to give microscopic interpretation to entropy formulas for more general theories than the Einstein theory. Here, the D-dimensional extended Gauss–Bonnet theory was considered. It was shown that using Carlip method [8] and asking certain boundary conditions near black hole horizon one can define an algebra of diffeomorphisms containing Virasoro algebra as its subalgebra.

Calculation of central charge enables, with the help of Cardy formula, to find the entropy which is as expected to be different than area law but agrees with Gauss–Bonnet entropy as derived in [12,13].

The result is more general then alternative derivation by us in Ref. [6], because here we do not have to restrict ourselves to spherical symmetry. Also here it is possible to calculate separately central charge and eigenvalue of Virasoro generator L_0. It is also encouraging that it shows that the two methods give consistent results.

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